Seminar 2 Solutions

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Roadmap

Exercise 1

Exercise 2

Unbiasedness

Definition

An estimator of a given paramater is said to be unbiased if its expected value is equal to the true value of the parameter:

$$\mathbb{E}[\hat{\theta}(\xi)] = \theta_0$$

OLS unbiasedness (Exercise 1)

- OLS Assumptions:
  1. Linear in parameters
  2. Random sampling
  3. No perfect collinearity
  4. Zero conditional mean: $E[u_i|x_i] = E[u_i] = 0.$
- Under assumptions 1-4 the OLS estimator is unbiased

$$E[\hat{\beta}] = \beta$$

OLS unbiasedness: proof

Variance of OLS estimator: derivation (Exercise 2)

5. Homoskedasticity: $Var(\mathbf{u}|\mathbf{X}) = \sigma^2\mathbf{I}_n \quad \sigma^2 > 0$

Variance of OLS estimates

- The variance of the OLS estimates is given by:

$$\text{Var}(\hat{\beta}|\mathbf{X}) = \sigma^2(\mathbf{X}'\mathbf{X})^{-1}$$

- The standard errors are given by:

$$se\left(\hat{\beta}_j\right) = \sqrt{\text{Var}\left(\hat{\beta}_j \mid \mathbf{X}\right)} = \sqrt{\sigma^2\left(\mathbf{X}'\mathbf{X}\right)^{-1}_{jj}} = \sigma\sqrt{\left(\mathbf{X}'\mathbf{X}\right)^{-1}_{jj}}.$$

- σ^2 is not observed. Obtain an unbiased estimate through the OLS residuals $\hat{\mathbf{u}} = \mathbf{y} - \mathbf{X}\hat{\beta}$

Variance of OLS estimates

- then

$$\hat{\sigma}^2 = \frac{\hat{\mathbf{u}}'\hat{\mathbf{u}}}{n-k-1},$$

- therefore,

$$se\left(\hat{\beta}_j\right) = \hat{\sigma}\sqrt{(\mathbf{X}'\mathbf{X})^{-1}_{jj}}.$$

- Increasing the sample size n reduces $\hat{\sigma}^2$ and hence the standard errors.

SE of OLS estimates: SLR and matrices (Exercise 5)

- In the SLR case, X is a $n \times 2$ matrix:

$$\mathbf{X} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}$$

- then $X'X$ is a 2×2 matrix:

$$\mathbf{X}'\mathbf{X} = \begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix}$$

- and its inverse is (see matrix algebra slides):

$$\left(\mathbf{X}'\mathbf{X}\right)^{-1} = \frac{1}{n \sum x_i^2 - \left(\sum x_i\right)^2} \begin{bmatrix} \sum x_i^2 & -\sum x_i \\ -\sum x_i & n \end{bmatrix}$$

SE of OLS estimates: SLR and matrices (Exercise 5)

- to compute the standard errors we use the formula:

$$se\left(\hat{\beta}_j\right) = \hat{\sigma}\sqrt{(\mathbf{X}'\mathbf{X})^{-1}_{jj}}.$$

- therefore, the standard errors for $\hat{\beta}_0$ and $\hat{\beta}_1$ are:

$$se\left(\hat{\beta}_0\right) = \hat{\sigma}\sqrt{\frac{\sum x_i^2}{n\sum x_i^2 - \left(\sum x_i\right)^2}}$$

$$se\left(\hat{\beta}_1\right) = \hat{\sigma}\sqrt{\frac{n}{n\sum x_i^2 - \left(\sum x_i\right)^2}}$$

Exercise 1

Standard Errors of OLS Estimates

Model:

$$Wage = \beta_0 + \beta_1 Educ + u$$

- $\hat{\beta}_1 = 1.04, \quad \hat{\beta}_0 = -6.90.$
- $\hat{\sigma}^2 = \frac{1}{n-2} \sum_{i=1}^{n} \hat{u}_i^2 = 10.31.$

$$\text{Var}(\hat{\beta}_0) = 19.54 \implies \text{se}(\hat{\beta}_0) = 4.42.$$

$$\text{Var}(\hat{\beta}_1) = 0.108 \implies \text{se}(\hat{\beta}_1) = 0.33.$$

Roadmap

Exercise 1

Exercise 2

Exercise 2.2-2.3

Collinearity

Model:

$$y_i = \beta_0 + \beta_1 x_{1,i} + \beta_2 x_{2,i} + \beta_3 x_{3,i} + u_i \quad i = 1, \ldots, n$$

- Collinearity: If $x_3 = x_1 + x_2 + 6$, perfect collinearity exists, making OLS infeasible.
- assumption 3 is violated.
- rank of $\mathbf{X}'\mathbf{X}$ is 2, not 3. Cannot invert the matrix.

Exercise 2.2-2.3

Irrelevant Variables

True model:

$$y_i = \beta_0 + \beta_1 x_{1,i} + \beta_2 x_{2,i} + u_i \quad i = 1, \ldots, n$$

Estimated model:

$$y_i = \beta_0 + \beta_1 x_{1,i} + \beta_2 x_{2,i} + \beta_3 x_{3,i} + u_i \quad i = 1, \ldots, n$$

- Irrelevant Variables: Including irrelevant variables (e.g., x_3) does not affect the unbiasedness but reduces efficiency.

Roadmap

Appendix

Appendix: Proof of OLS Unbiasedness

1.1. EX1

① Linear in parameters $y = \beta_0 + \beta_1 x + u$

② Random sampling

③ No multicollinearity

④ Errors have zero conditional means $E[u_i | x_i] = 0 \Rightarrow E[u_i] = 0$

under 1-4 OLS estimates are unbiased: $E[\hat{\beta}_0] = \beta_0$, $E[\hat{\beta}_1] = \beta_1$

$$\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y}) = \frac{1}{n}\sum_{i=1}^{n}(x_i y_i - \bar{y}x_i - \bar{x}y_i + \bar{x}\bar{y})$$

$$= \frac{1}{n}\sum_{i=1}^{n}(x_i y_i) - \bar{y}\sum_{i=1}^{n}\frac{x_i}{n} - \bar{x}\sum_{i=1}^{n}\frac{y_i}{n} + \frac{n}{n}\bar{x}\bar{y} = \frac{1}{n}\sum_{i=1}^{n} x_i y_i - \bar{x}\bar{y}$$

$$= \frac{1}{n}\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n}\frac{x_i}{n}\sum_{i=1}^{n}\frac{y_i}{n} = \frac{1}{n}\sum_{i=1}^{n} x_i y_i - \frac{1}{n^2}\sum_{i=1}^{n} x_i y_i$$

$$= \frac{1}{n}\left[\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} y_i \bar{x}\right]$$

$$= \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})y_i$$

$$\bar{x} = \sum_{i=1}^{n}\frac{x_i}{n}$$

$$* \sum_{i=1}^{n}(x_i-\bar{x}) = \sum_{i=1}^{n} x_i - \sum_{i=1}^{n}\bar{x}$$

$$= \bar{x}n - \bar{x}n = 0$$

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{\sum_{i=1}^{n}(x_i-\bar{x})^2}$$

$$= \frac{\sum_{i=1}^{n}(x_i-\bar{x})y_i}{\sum_{i=1}^{n}(x_i-\bar{x})^2}$$

using ① $y_i = \beta_0 + \beta_1 x_i + u_i$

$$= \frac{\sum_{i=1}^{n}(x_i-\bar{x})(\beta_0+\beta_1 x_i + u_i)}{\sum_{i=1}^{n}(x_i-\bar{x})^2} = \frac{\beta_0 \overbrace{\sum_{i=1}^{n}(x_i-\bar{x})}^{*0} + \beta_1 \overbrace{\sum_{i=1}^{n}(x_i-\bar{x})x_i}^{\sum_{i=1}^{n}(x_i-\bar{x})^2} + \sum_{i=1}^{n}(x_i-\bar{x})u_i}{\sum_{i=1}^{n}(x_i-\bar{x})^2}$$

$$= \frac{\beta_0 \times 0}{\sum_{i=1}^{n}(x_i-\bar{x})^2} + \beta_1 \times \frac{\cancel{\sum_{i=1}^{n}(x_i-\bar{x})^2}}{\cancel{\sum_{i=1}^{n}(x_i-\bar{x})^2}} + \frac{\sum_{i=1}^{n}(x_i-\bar{x})u_i}{\sum_{i=1}^{n}(x_i-\bar{x})^2} = \beta_1 + \frac{\sum_{i=1}^{n}(x_i-\bar{x})u_i}{\sum_{i=1}^{n}(x_i-\bar{x})^2}$$

$$E[\hat{\beta}_1] = E\left[\beta_1 + \frac{\sum_{i=1}^{n}(x_i-\bar{x})u_i}{\sum^{n}(x_i-\bar{x})^2}\right] = \beta_1 + \frac{\sum_{i=1}^{n}(x_i-\bar{x})}{\sum^{n}(x_i-\bar{x})^2}\underbrace{E[u_i]}_{0 \text{ Using 4}} \Rightarrow E[\hat{\beta}_1] = \beta_1$$

Appendix: Variance of OLS Estimator

$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$ using ① $y_i = \beta_0 + \beta_1 x_i + u_i$ $\bar{y} = E[y_i] = \beta_0 + E[\beta_1 x_i] + \overset{=0}{E[u_i]}$ $E[\cancel{\beta_1}] =$

$= \beta_0 + \beta_1 \bar{x} - \hat{\beta}_1 \bar{x} = \beta_0 + \bar{x}(\beta_1 - \hat{\beta}_1)$

$= \beta_0 + E[\beta_1] \bar{x}$

$\overset{!}{=} \beta_0 + \beta_1 \bar{x}$

$E[\beta_0] = E[\beta_0 + \bar{x}(\beta_1 - \hat{\beta}_1)] = E[\beta_0] + \bar{x} \underbrace{E[\beta_1 - \hat{\beta}_1]}_{0} = \beta_0$

EX2

$\underset{n\times 1}{y} = \underset{n\times 4}{X} \underset{4\times 1}{\beta} + \underset{n\times 1}{u}$

⑤ Homoskedasticity: $Var(u|X) = \sigma^2 I_n$ $\sigma^2 > 0$

OLS $\hat{\beta} = (X'X)^{-1} X'y$ ①

$= (X'X)^{-1} X'(X\beta + u) = \cancel{(X'X)^{-1} X'X} \beta + (X'X)^{-1} X'u$

unbiasedness

$E[\hat{\beta}] = E[\cancel{(X'X)^{-1} X'X} \beta + (X'X)^{-1} X'u] = E[\beta] + E[(X'X)^{-1} X'u]$

$= \beta + (X'X)^{-1} X' E[\underbrace{u}_{0}]$

$E[\hat{\beta}] = \beta$ $\overset{!}{=} \beta$

variance of $\hat{\beta}$

$var(\hat{\beta}|X) = var[(X'X)^{-1} X'y|X] = var[(\beta + (X'X)^{-1} X'u)|X] = (X'X)^{-1} X' \overbrace{var(u|X)}^{⑤ \nearrow \sigma^2 I_n} X (X'X)^{-1}$

$= \sigma^2 \cancel{(X'X)^{-1} X'X} (X'X)^{-1} = \sigma^2 (X'X)^{-1}$