Seminar 5 Solutions

Stationarity and Time Series Properties

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Roadmap

- Exercise 1: True/False
  - Gauss-Markov Assumptions
  - Trending Variables
  - Seasonality
- Exercise 3: Stationarity
  - Stochastic Processes
  - Weakly Stationary Process
- Exercise 4: Stationarity
  - Setup
  - Expected Value and Variance
  - Autocovariance
  - Autocorrelation
- Exercise 6: Stock Return Predictability

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Disclaimer

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Full solutions are available on my.wbs. All exercises are examinable material, not just the ones we covered in the seminars.

Exercise 1

Gauss-Markov Assumptions

Q: The OLS estimator in a time-series setting is unbiased under the first three Gauss-Markov assumptions.

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Gauss-Markov Assumptions

Q: The OLS estimator in a time-series setting is unbiased under the first three Gauss-Markov assumptions.

1. TS.1: Linearity in parameters
2. TS.2: No perfect collinearity
3. TS.3: Strict exogeneity / Zero conditional mean

When we add the following two assumptions, the OLS estimator is also BLUE.

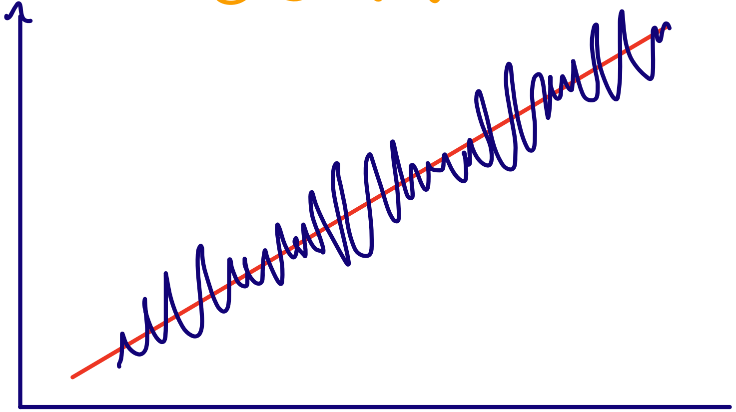
4. TS.4: Homoskedasticity
5. TS.5: No serial correlation

Exercise 1

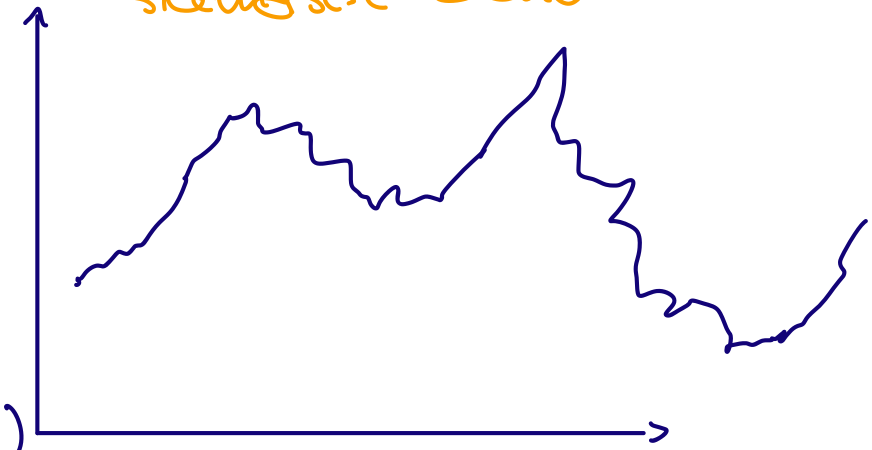
Trending Variables

Q: A trending variable cannot be used as a dependent variable in the multiple linear regression model.

deterministic trend



stochastic trend



I(1)

$$Y_t = \beta_0 + \beta_1 X_t + \underline{\underline{\alpha}}\, t + u_t$$

$\rightarrow$ RW: $\underline{X_t} = X_{t-1} + \varepsilon_t \quad \varepsilon_t \sim N(0, \sigma^2_\varepsilon)$

$$\Delta x_t = x_t - x_{t-1}$$

I(0) $\Delta x_t = \varepsilon_t$

Exercise 1

Trending Variables

Q: A trending variable cannot be used as a dependent variable in the multiple linear regression model.

- Trending variables *can* be used as dependent variables in a linear regression model.
- However, be cautious when interpreting the results:
  - spurious relationship between y_t and trending explanatory variables.
- Including a time trend in the regression is advisable when dependent and/or independent variables are trending.
- The usual R^2 measure can be misleading when the dependent variable is trending.

Exercise 1

Seasonality

Q: Seasonality is not an issue when using annual time-series observations.

Exercise 1

Seasonality

Q: Seasonality is not an issue when using annual time-series observations.

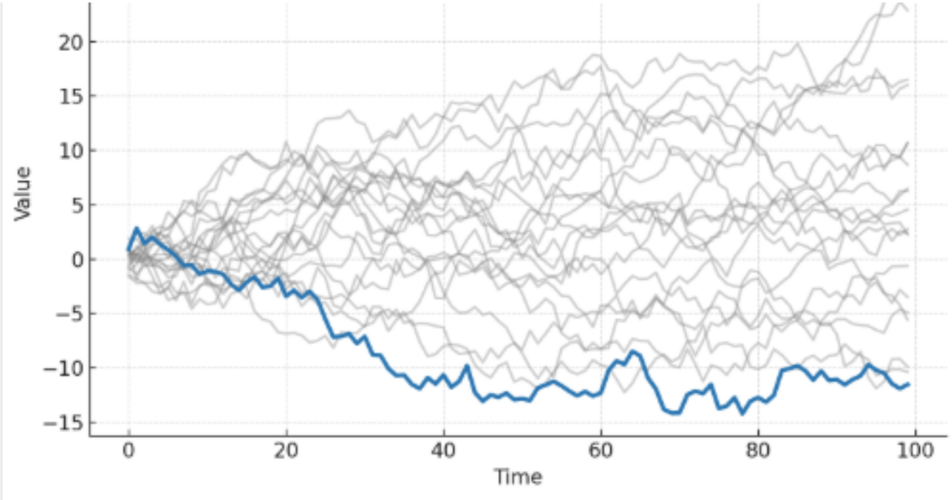
- Each period represents a year and this is not associated with any season.

Roadmap

Exercise 3

Stochastic Processes

We model time series as a sequence of **random variables**. A collection of random variables indexed by time is called a **stochastic process**



- We observe only one realization of the process
- But all the grey paths were equally likely to have happened
- We want to make inference about the **whole process**, and we see only one path

Exercise 3

Weakly Stationary Process: Correlation

Let $\{x_t : t = 1, 2, \ldots, T\}$ be a weakly stationary process.

Define $\gamma_h = \mathrm{Cov}(x_t, x_{t+h})$ for $h \geq 0$. Then $\gamma_0 = \mathrm{Var}(x_t)$. Show that

$$\mathrm{Corr}(x_t, x_{t+h}) = \frac{\gamma_h}{\gamma_0}.$$

Weak (or covariance) Stationarity

A stochastic process $\{x_t : t = 1, 2, \ldots\}$ is said to be weakly stationary if:

$$\mathbb{E}(x_t) = \mu, \quad \mathrm{Var}(x_t) = \sigma^2, \quad \mathrm{Cov}(x_t, x_{t+h}) = f(h).$$

X_{1980} X_{1985} X_{2015} X_{2020}

A weakly stationary process is uniquely determined by its mean, variance, and autocovariance function.

Exercise 3

$$\mathrm{corr}(X,Y) = \frac{\mathrm{cov}(X,Y)}{\sigma_X \, \sigma_Y}$$

Derivation

$$\gamma_h = \mathrm{cov}(x_t, x_{t+h})$$

$h=0 \quad \underline{\gamma_0} = \mathrm{cov}(x_t, x_{t+0}) = \mathrm{cov}(x_t, x_t) = \mathrm{var}(x_t)$

$$\mathrm{corr}(x_t, x_{t+h}) = \frac{\overbrace{\mathrm{cov}(x_t, x_{t+h})}^{\gamma_h}}{\underbrace{\sqrt{\mathrm{var}(x_t)}}_{\sqrt{\gamma_0}} \, \underbrace{\sqrt{\mathrm{var}(x_{t+h})}}_{\sqrt{\gamma_0}}} = \frac{\gamma_h}{\gamma_0}$$

$$\mathrm{var}(x_t) = \mathrm{var}(x_{t+1}) = \ldots = \mathrm{var}(x_{t+h}) = \gamma_0$$

Roadmap

Exercise 4

Setup

Suppose that a time-series process $\{x_t : t = 1, 2, \ldots, T\}$ is given by

$$x_t = z + \epsilon_t,$$

for all $t = 1, 2, \ldots, T$, where ϵ_t is an i.i.d. sequence with mean zero and variance σ_ϵ^2. The random variable z is constant over time, and it has mean zero and variance σ_z^2. Furthermore, assume that ϵ_t is uncorrelated with z.

$\longrightarrow \varepsilon_t \overset{iid}{\sim} (0, \sigma_\varepsilon^2)$

$z \sim (0, \sigma_z^2)$

$\text{cov}(\varepsilon_t, z) = 0$

Exercise 4

Expected Value and Variance

$x_t = z + \varepsilon_t$

Q: Find the expected value and variance of x_t. Do your answers depend on t?

$E[x_t] = E[z + \varepsilon_t] = \underbrace{E[z]}_{=0} + \underbrace{E[\varepsilon_t]}_{=0}$

$E[x_{t+u}] = 0$

$= 0 \leftarrow \forall t$

$var(x_t) = var(z + \varepsilon_t) = var(z) + var(\varepsilon_t) + \underbrace{2cov(z, \varepsilon_t)}_{=0}$

$= \sigma_z^2 + \sigma_\varepsilon^2 \quad \forall t$

$cov(\varepsilon_t, z) = \underbrace{E[\varepsilon_t z]}$

$E[x_t] = 0$

Exercise 4

Autocovariance

$$\operatorname{cov}(X,Y) = \underbrace{\boxed{E[XY]} - \cancel{E[X]E[Y]}}_{} \qquad E[\varepsilon_t] = 0$$

Q: Find $\operatorname{Cov}(x_t, x_{t+h})$ for any t and h. Is x_t a weakly stationary process?

$$\operatorname{cov}(x_t, x_{t+h}) = E[x_t x_{t+h}] - \underbrace{E[x_t]}_{=0}\underbrace{E[x_{t+h}]}_{=0}$$

$$= E[x_t x_{t+h}]$$

$$= E[(z+\varepsilon_t)(z+\varepsilon_{t+h})]$$

$h=1$

$$\operatorname{cov}(x_t, x_{t+1}) = E[(z+\varepsilon_t)(z+\varepsilon_{t+1})]$$

$$= E[z^2 + \varepsilon_t z + z\varepsilon_{t+1} + \varepsilon_t\varepsilon_{t+1}]$$

$$= E[z^2] + \underbrace{E[\varepsilon_t z]}_{=0} + E[z\varepsilon_{t+1}] + E[\varepsilon_t \varepsilon_{t+1}]$$

$\operatorname{cov}(\varepsilon_t, z)$ $\quad 0 \quad 0$

$$= E[z^2]$$

$h=2$

Exercise 4

Autocorrelation

$= \sigma_z^2 \quad \forall t \quad \equiv \text{constant}$

Q: Show that $\mathrm{Corr}(x_t, x_{t+h}) = \frac{\sigma_z^2}{\sigma_z^2+\sigma_\epsilon^2}$ for any t and h.

$$\mathrm{cov}(x_t, x_{t+h}) = \frac{1}{h} \sigma_z^2$$

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Exercise 6: Stock Return Predictability

Exercise 6

Predictive Model – Regression Output

Model:

$$\text{Return}_t = \beta_0 + \beta_1 \,\text{Return}_{t-1} + u_t, \quad u_t \sim N(0, \sigma^2).$$

Source	SS	df	MS
Model	10.6866231	1	10.6866231
Residual	3059.73817	687	4.45376735
Total	3070.42479	688	4.46282673

Number of obs	= 689
F(1, 687)	= 2.40
Prob > F	= 0.1218
R-squared	= 0.0035
Adj R-squared	= 0.0020
Root MSE	= 2.1104

return	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
return_1	.0588984	.0380231	1.55	0.122	-.0157569	.1335538
_cons	.179634	.0807419	2.22	0.026	.0211034	.3381646

Q: Compute the unconditional mean and variance of the returns.

Exercise 6

Unconditional Mean and Variance

- AR(1) model: $\mathrm{Return}_t = \beta_0 + \beta_1 \, \mathrm{Return}_{t-1} + u_t$.
- Unconditional mean:

$$E(\mathrm{Return}_t) = \frac{\beta_0}{1-\beta_1} = \frac{0.179}{1-0.058} = 0.19.$$

- Unconditional variance of returns is

$$\mathrm{Var}(\mathrm{Return}_t) = \frac{\sigma^2}{1-\beta_1^2} = \frac{4.45}{1-0.058^2} = 4.465.$$

- where the error variance is computed from the residuals as

$$\hat{\sigma}^2 = \frac{\hat{u}'\hat{u}}{T-k-1} = \frac{3059}{687} = 4.45.$$

Roadmap

Exercise 1

Expectations Hypothesis – Setup

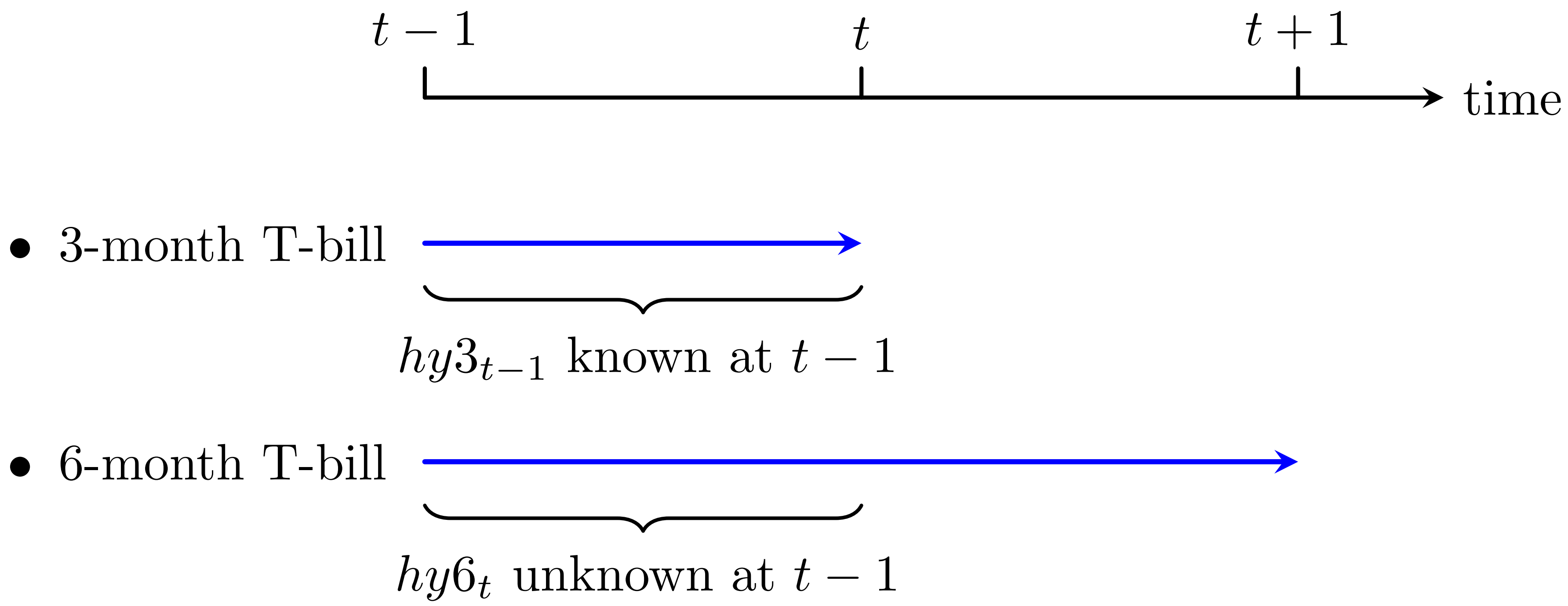
Let $hy6_t$ denote the three-month holding-period yield (in percent) from buying a six-month T-bill at time $(t-1)$ and selling it at time t (three months later) as a three-month T-bill. Let $hy3_{t-1}$ be the three-month holding-period yield from buying a three-month T-bill at time $(t-1)$.

An implication of the EH is that no other variable dated as $t-1$ or earlier should help explain $hy6_t$ once $hy3_{t-1}$ has been controlled for. Including one lag of the spread between the 6-month and 3-month T-bills rates yields

$$hy6_t = \beta_0 + \beta_1 hy3_{t-1} + \beta_2 (r6_{t-1} - r3_{t-1}) + u_t$$

Exercise 1

Visual Representation



Exercise 1

Estimation Results

Source	SS	df	MS
Model	86.8264477	2	43.4132238
Residual	11.2905482	120	.094087901
Total	98.1169958	122	.804237671

Number of obs	= 123
F(2, 120)	= 461.41
Prob > F	= 0.0000
R-squared	= 0.8849
Adj R-squared	= 0.8830
Root MSE	= .30674

hy6	Coef.	Std. Err.	t	P>|t|	[95% Conf. Interval]
hy3_1	1.053483	.0385001	27.36	0.000	
spr63_1	.4799728	.1085678	4.42	0.000	
_cons	-.1229242	.066781	-1.84	0.068	

Q: Is β_1 statistically different from one?

Exercise 1

Computing the t-Statistic

- The OLS estimate of β_1 is $\hat{\beta}_1 = 1.053$.
- Compute the t-statistic for testing $H_0 : \beta_1 = 1$:

$$t_{\hat{\beta}_1} = \frac{\hat{\beta}_1 - 1}{\text{se}(\hat{\beta}_1)} = \frac{1.053 - 1}{0.038} = \frac{0.053}{0.038} = 1.39.$$

- Since $1.39 < c$, fail to reject H_0: the estimate is not statistically different from 1 at usual significance levels.

Exercise 1

Lagged Spread Significance

Q: Is the lagged spread term significant? Should we invest in six-month or three-month T-bills if $r6$ is above $r3$ at time $t-1$?

- The lagged spread is clearly significant, as the p-value is 0.000.
- Therefore, we reject the null hypothesis $H_0 : \beta_2 = 0$ at the 1% significance level.
- For the buying strategy when $r_6 > r_3$, the three-month holding-period yield from buying a six-month T-bill is higher than what is implied by the Expectations Hypothesis (EH).
- As a result, it is more favorable to invest in a six-month T-bill.

Exercise 1

Time Trend

Suppose you estimate a linear time trend model on the three-month holding-period yield from buying a six-month T-bill:

$$hy6_t = \alpha_0 + \alpha_1 t + u_t$$

Source	SS	df	MS
Model	34.8025748	1	34.8025748
Residual	63.314421	121	.523259678
Total	98.1169958	122	.804237671

Number of obs	=	123
F(1, 121)	=	66.51
Prob > F	=	0.0000
R-squared	=	0.3547
Adj R-squared	=	0.3494
Root MSE	=	.72337

hy6	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
t	.0149814	.001837	8.16	0.000	.0113446	.0186182
_cons	.770868	.1328443	5.80	0.000	.5078677	1.033868

- We can clearly reject the null hypothesis $H_0 : \alpha_1 = 0$ (p-value = 0.000).