Seminar 7 Solutions

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Disclaimer

Full solutions are available on my.wbs. All exercises are examinable material, not just the ones we covered in the seminars.

Roadmap

Exercise 1

Exercise 2

Exercise 3

Exercise 4

Exercise 1: Simple Linear Regression (SLR) Model

Consider the savings function:

$$sav = \beta_0 + \beta_1 inc + u, \quad u = \sqrt{inc} \cdot e \quad \mathrm{E}[e] = 0 \quad \mathrm{Var}[e] = \sigma_e^2$$

Assume that e is independent of inc. $e \perp\!\!\!\perp inc$

Show that the zero conditional mean assumption is satisfied.

$$E[u \mid x] = 0$$

$$E[u \mid inc] = 0$$

$$= E[\sqrt{inc} \cdot e \mid inc]$$

$$= \sqrt{inc}\, E[e \mid inc]$$

$$= \sqrt{inc}\, E[e] = \sqrt{inc} \cdot 0 = 0$$

$$E[e] = 0$$

Homoskedasticity

↳ $var(u|x) = \sigma^2 \quad \sigma^2 > 0$

$var(u|inc) = var(\sqrt{inc}\, e \,|\, inc)$

$= inc\; var(e|inc)$

$= inc\; var(e) = inc\, \sigma_e^2$

Derivation

- When we condition on inc, $\sqrt{inc}$ becomes a constant.
- Therefore:

$$E[u|inc] = E[\sqrt{inc} \cdot e|inc] = \sqrt{inc}E[e|inc]$$

- Since $E[e|inc] = E[e] = 0$, it follows that:

$$E[u|inc] = 0.$$

Exercise 1: Violation of Homoskedasticity

Show that the homoskedasticity assumption SLR.5 is violated; that is, $\text{Var}(u|inc) = \sigma_e^2 inc$.

- Computing the conditional variance:

$$\text{Var}[u|inc] = \text{Var}[\sqrt{inc} \cdot e|inc]$$

- Since variance operators allow constants to be factored out:

$$\begin{aligned}\text{Var}[u|inc] &= inc \cdot \text{Var}[e|inc] \\ &= inc \cdot \sigma_e^2.\end{aligned}$$

- This shows that variance depends on inc, violating the homoskedasticity assumption.

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Exercise 2: Multiple Linear Regression (MLR) Model

$\Rightarrow study + sleep + work + leisure = 168$

$study = 168 - sleep - work - leisure$

Model

$$GPA = \beta_0 + \beta_1 study + \beta_2 sleep + \beta_3 work + \beta_4 leisure + u$$

does it make sense to hold *sleep*, *work*, and *leisure* fixed, while changing *study*?

- No. By definition, $study + sleep + work + leisure = 168$.
- If we change $study$, at least one other category must also change to maintain the total sum.

Violations of Gauss-Markov Assumptions

- $study$ is a perfect linear combination of other regressors. This holds for every observation, violating MLR.3.

$$study = 168 - sleep - work - leisure.$$

How to fix this?

- Drop one of the independent variables, say $leisure$:

$$GPA = \beta_0 + \beta_1 study + \beta_2 sleep + \beta_3 work + u.$$

- β_1: change in GPA when $study$ increases by one hour, holding $sleep$, $work$, and u constant.
- If $sleep$ and $work$ are fixed but $study \uparrow$, then $leisure \downarrow$ by one hour.

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Dickey Fuller Test

r_{3t} has a unit root $\Rightarrow$ r_{3t} non-stationary

AR(1)

$$r_{3t} = \mu + \rho\, r_{3t-1} + \varepsilon_t \qquad \varepsilon_t \sim N(0, \sigma_\varepsilon^2)$$

$|\rho| = 1 \Rightarrow$ RW, nonstationary

$H_0: \; |\rho| = 1$ non stationarity

$H_1: \; |\rho| < 1$ stationarity

$$r_{3t} - r_{3t-1} = \mu + (\rho - 1) r_{3t-1} + \varepsilon_t$$

$\gamma_1 \Delta r_{t-1} + \gamma_2 \Delta r_{t-2} \dots$

$$\Rightarrow \Delta r_{3t} = \mu + \theta\, r_{3t-1} + \varepsilon_t$$

Δr_{3t}: $I(0)$

$H_0: \; \theta = 0$ RW

$H_1: \; \theta < 0$ stationarity

$t_\theta < DF_{cv} \Rightarrow$ reject

H_0: r_{3t} $I(1)$

$\Delta r_{3t} \Rightarrow I(0)$

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Exercise 4: Error Correction Model

Suppose that the process $\{(x_t, y_t) : t = 0, 1, 2, \dots\}$ satisfies the equations:

$$y_t = \beta x_t + u_t$$

and

$$\Delta x_t = \gamma \Delta x_{t-1} + \nu_t,$$

where:

- $E[u_t | I_{t-1}] = E[\nu_t | I_{t-1}] = 0$,
- I_{t-1} contains information on x and y dated at time $t-1$ and earlier,
- $\beta \neq 0$, and $|\gamma| < 1$ (so that x_t, and therefore y_t, is $I(1)$).

Exercise 4: Error Correction Model

Show that the given equations imply an error correction model:

$$\Delta y_t = \gamma_1 \Delta x_{t-1} + \delta(y_{t-1} - \beta x_{t-1}) + e_t,$$

where:

$$\gamma_1 = \beta\gamma, \quad \delta = -1, \quad e_t = \beta\nu_t + u_t.$$

Step 1: Expressing Δy_t

Start by subtracting y_{t-1} from both sides of $y_t = \beta x_t + u_t$:

I(0)

$$\Delta y_t = \beta x_t - y_{t-1} + u_t.$$

$$= \beta x_t - \beta x_{t-1} + \beta x_{t-1} - y_{t-1} + u_t$$

$$\Delta y_t = \beta \Delta x_t - y_{t-1} + u_t$$

I(0) I(0)

$$\Delta x_t = \gamma \Delta x_{t-1} + v_t$$

Step 2: Introducing βx_{t-1}

- Add and subtract βx_{t-1} on the right-hand side:

$$\Delta y_t = \beta \Delta x_t - (y_{t-1} - \beta x_{t-1}) + u_t.$$

- This isolates the error correction term $y_{t-1} - \beta x_{t-1}$.

Now substitute $\Delta x_t = \gamma \Delta x_{t-1} + \nu_t$ into the equation:

$$\Delta y_t = \beta(\gamma \Delta x_{t-1} + \nu_t) - (y_{t-1} - \beta x_{t-1}) + u_t.$$

Step 3: Final Expression

- Distribute β in the equation:

$$\Delta y_t = \beta\gamma\Delta x_{t-1} - (y_{t-1} - \beta x_{t-1}) + (\beta\nu_t + u_t).$$

- Recognizing the key components:

$$\gamma_1 = \beta\gamma, \quad \delta = -1, \quad e_t = \beta\nu_t + u_t.$$

- Thus, we obtain the final error correction model:

$$\Delta y_t = \gamma_1\Delta x_{t-1} + \delta(y_{t-1} - \beta x_{t-1}) + e_t.$$

$X_t \in I(1)$

$Y_t \in I(1)$

$$Y_t = \alpha + \beta X_t + u_t$$

$$u_t = Y_t - \alpha - \beta X_t$$

$\downarrow$

DF

$$\Delta y_t = \alpha + \beta \Delta x_t$$