Seminar 8 Solutions

Giulio Rossetti*

giuliorossetti94.github.io

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* email: giulio.rossetti.1@wbs.ac.uk

Disclaimer

Full solutions are available on my.wbs. All exercises are examinable material, not just the ones we covered in the seminars.

Roadmap

Question 1: Forecasting with ARIMA(1,2)

Write down a set of equations to produce one-step, two-step, and three-step ahead forecasts for y_t, given that it follows an ARIMA(1,2) process:

$$y_t = \underbrace{\phi y_{t-1}}_{\text{AR(1)}} + \varepsilon_t + \underbrace{\theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2}}_{\text{MA(2)}}, \quad \varepsilon_t \sim N(0, \sigma_t^2).$$

Roadmap

$$y_t = \phi y_{t-1} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} \qquad \varepsilon_t \sim N(0, \sigma_\varepsilon^2)$$

$t+1$

$$y_{t+1} = \phi y_t + \varepsilon_{t+1} + \theta_1 \varepsilon_t + \theta_2 \varepsilon_{t-1} \leftarrow$$

$t+2$

$$y_{t+2} = \phi y_{t+1} + \varepsilon_{t+2} + \theta_1 \varepsilon_{t+1} + \theta_2 \varepsilon_t \leftarrow$$

$t+3$

$$y_{t+3} = \phi y_{t+2} + \varepsilon_{t+3} + \theta_1 \varepsilon_{t+2} + \theta_2 \varepsilon_{t+1}$$

FORECAST

$$E[y_{t+1} \mid I_t] = E_t[y_{t+1}]$$

$$= E_t[\phi y_t + \varepsilon_{t+1} + \theta_1 \varepsilon_t + \theta_2 \varepsilon_{t-1}]$$

$$= \phi E_t[y_t] + \underbrace{E_t[\varepsilon_{t+1}]}_{0} + \theta_1 E_t[\varepsilon_t] + \theta_2 E_t[\varepsilon_{t-1}]$$

$$= \phi y_t + \theta_1 \varepsilon_t + \theta_2 \varepsilon_{t-1}$$

$$E_t[y_{t+2}] = E_t[\phi y_{t+1} + \varepsilon_{t+2} + \theta_1 \varepsilon_{t+1} + \theta_2 \varepsilon_t]$$

$$= \phi E_t[y_{t+1}] + \theta_2 \varepsilon_t$$

$$= \phi(\phi y_t + \theta_1 \varepsilon_t + \theta_2 \varepsilon_{t-1}) + \theta_2 \varepsilon_t = \phi^2 y_t + \phi \theta_1 \varepsilon_t + \phi \theta_2 \varepsilon_{t-1} \ldots$$

$$\ldots + \theta_2 \varepsilon_t$$

$$= \phi^2 y_t + \varepsilon_t(\phi\theta_1 + \theta_2) + \phi\theta_2 \varepsilon_{t-1}$$

$$E_t[y_{t+3}] = E_t[\phi y_{t+2} + \varepsilon_{t+3} + \theta_1 \varepsilon_{t+2} + \theta_2 \varepsilon_{t+1}]$$

$$= \phi E_t[y_{t+2}]$$

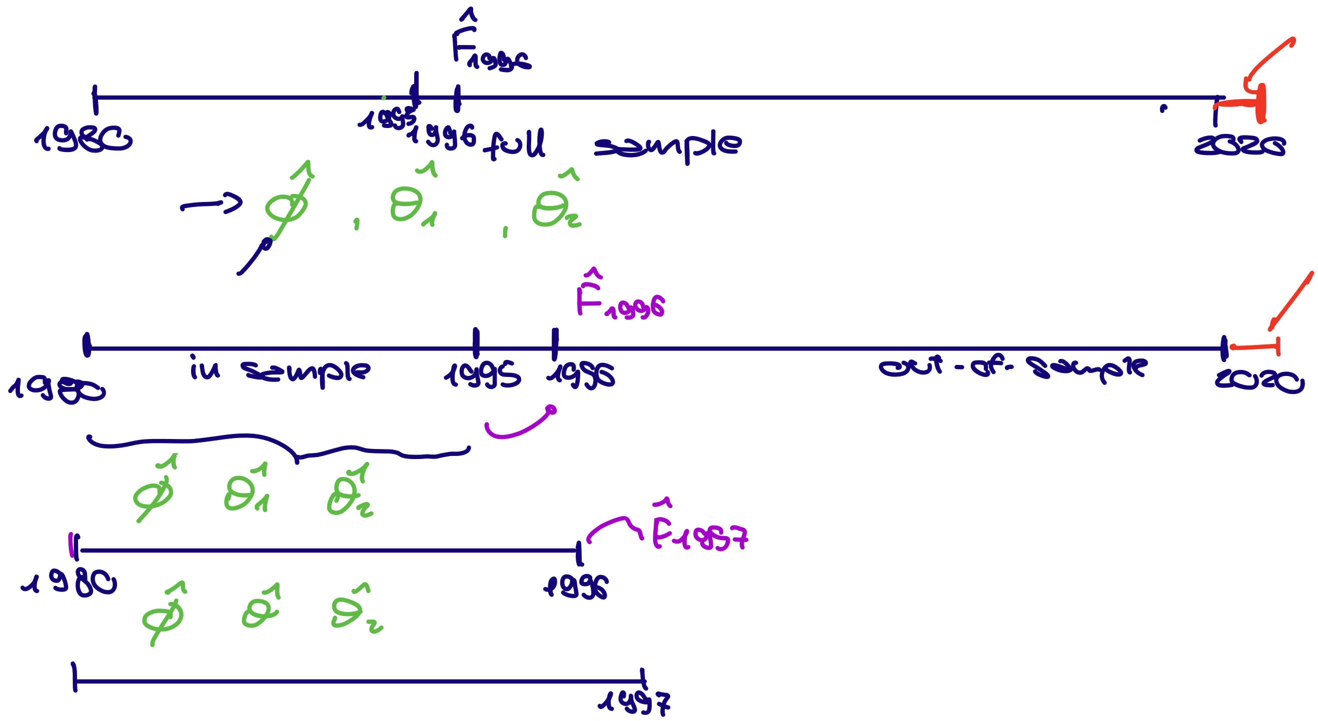
Question 2: Pseudo Out-of-Sample Forecasting

Outline the stages of a pseudo out-of-sample forecasting evaluation, comparing a ”benchmark” AR(1) with the ARIMA(1,2) model.

Hints:

- (i) Recursive/repeated estimation with an expanding sample.
- (ii) Choice of h-steps forecasting at each point.
- (iii) Forecast error comparison via root mean square forecast error (RMSE) and statistical testing.

ARIMA(1,2) $y_t = \phi y_{t-1} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2}$ $\quad \varepsilon_t \sim N(0, \sigma_\varepsilon^2)$



ARIMA(1,2) AR(1)

$\hat{F}_{1996}$ $\quad F_{1996}$ $\quad e_{1996} = F_{1996} - \hat{F}_{1996}$ $\quad \tilde{e}_{1996}$

$\vdots \qquad \vdots \qquad \vdots$

$\hat{F}_{2020}$ $\qquad \tilde{e}_{2020}$

$$MSE = \frac{1}{T} \sum e_t^2 \qquad \widetilde{MSE} = \frac{1}{T} \sum \tilde{e}_t$$

$$RMSE_{ARIMA} = \sqrt{MSE} \qquad RMSE_{AR} = \sqrt{\widetilde{MSE}}$$

$$I = \frac{RMSE_{ARIMA}}{RMSE_{AR}}$$

Roadmap

Question 3: Forecast Uncertainty in MA(2)

would you evaluate forecast uncertainty of 1-step and 2-step ahead forecasts?

Assume the forecasts come from an MA(2) process.

Hint: Consider the variances of the forecasts at each step.

$$y_t = \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} \qquad \varepsilon_t \sim N(0, \sigma^2_\varepsilon)$$

$t+1$

$$y_{t+1} = \varepsilon_{t+1} + \theta_1 \varepsilon_t + \theta_2 \varepsilon_{t-1} \quad A$$

$$E_t[y_{t+1}] = \theta_1 \varepsilon_t + \theta_2 \varepsilon_{t-1} \quad B$$

$t+2$

$$y_{t+2} = \varepsilon_{t+2} + \theta_1 \varepsilon_{t+1} + \theta_2 \varepsilon_t$$

$$E_t[y_{t+2}] = \theta_2 \varepsilon_t$$

$t+3$

$$y_{t+3} = \varepsilon_{t+3} + \theta_1 \varepsilon_{t+2} + \theta_2 \varepsilon_{t+1}$$

$$E_t[y_{t+3}] = 0$$

$$E_t[y_{t+h}] = 0 \quad h > 2$$

FORECAST ERROR

$$e_{t+1,t} = y_{t+1} - E_t[y_{t+1}] = \varepsilon_{t+1}$$

$$e_{t+2,t} = y_{t+2} - E_t[y_{t+2}] = \varepsilon_{t+2} + \theta_1 \varepsilon_{t+1}$$

$$e_{t+3,t} = y_{t+3} - E_t[y_{t+3}] = \varepsilon_{t+3} + \theta_1 \varepsilon_{t+2} + \theta_2 \varepsilon_{t+1}$$

VARIANCE FE

$$var(e_{t+1,t}) = var(\varepsilon_{t+1}) = \sigma^2_\varepsilon$$

$$var(e_{t+2,t}) = var(\varepsilon_{t+2} + \theta_1 \varepsilon_{t+1})$$

$$= \sigma^2_\varepsilon + \theta_1^2 \sigma^2_\varepsilon \quad + 2\,cov(\varepsilon_{t+2}, \theta_1 \varepsilon_{t+1}) = 0$$

$$= \sigma^2_\varepsilon (1 + \theta_1^2)$$

$$var(e_{t+3,t}) = var(\varepsilon_{t+3} + \theta_1 \varepsilon_{t+2} + \theta_2 \varepsilon_{t+1})$$

$$= \sigma^2_\varepsilon + \theta_1^2 \sigma^2_\varepsilon + \theta_2^2 \sigma^2_\varepsilon$$

$$\overset{!}{=} \sigma_\varepsilon^2 (1 + \theta_1^2 + \theta_2^2)$$